HECKE EIGENVALUES AND RELATIONS FOR DEGREE n SIEGEL EISENSTEIN SERIES OF SQUARE-FREE LEVEL

Lynne H. Walling

Abstract. We describe a basis of Siegel Eisenstein series of degree n, square-free level $\mathcal{N}$ and arbitrary character χ; then, without using knowledge of their Fourier coefficients, we evaluate the action of the Hecke operators $T(q)$, $T_j(q^2)$ $(1 \le j \le n)$ for primes $q|\mathcal{N}$. We find the space of Siegel Eisenstein series with square-free level has a basis of simultaneous eigenforms for these operators, and we compute the eigenvalues, thereby obtaining a multiplicity-one result. We then compute the action of the Hecke operators $T(p)$, $T_j(p^2)$ on a basis of Siegel Eisenstein series of level $\mathcal{N} \in \mathbb{Z}_+$ provided $4 \nmid \mathcal{N}$ and p is a prime with $p \nmid \mathcal{N}$, and from this construct a basis of simultaneous eigenforms.

§1. Introduction

Remark that space of Eisenstein series is invariant under Hecke operators
DEFINE:
Γ_∞^+
Refer to notation $\mathcal{E}_k^{(n)}(\mathcal{N},\chi)$

§2. Defining Siegel Eisenstein series

For $k, n, \mathcal{N} \in \mathbb{Z}_+$ and χ a character modulo $\mathcal{N}$, we want to define a degree n, weight k, level $\mathcal{N}$ Eisenstein series with character χ for each element of the quotient $\Gamma_\infty \backslash Sp_n(\mathbb{Z})/\Gamma_0(\mathcal{N})$. Given $\gamma_\rho \in Sp_n(\mathbb{Z})$, the natural object to define is

$$\mathbb{E}_\rho(\tau) = \sum_\gamma \overline{\chi}(\gamma)\, 1(\tau)|\gamma_\rho\gamma$$

where $\gamma \in \Gamma_0(\mathcal{N})$ varies so that $\Gamma_\infty\gamma_\rho\gamma$ varies over the (distinct) elements of $\Gamma_\infty\gamma_\rho\Gamma_0(\mathcal{N})$, and

$$1(\tau)|\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(C\tau + D)^{-k}$$

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for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$. If well-defined, this series converges absolutely uniformly on compact subsets of $\mathcal{H}_{(n)}$ provided $k \ge n+2$ (and hence is analytic).

[?? it is majorised by the level 1 Eisenstein series in the case k is even; what about when k is odd??]

Hence we assume $k \ge n+2$. However, defined as above, $\mathbb{E}_\rho$ may not be well-defined. Thus we over-sum, producing a well-defined function $\mathbb{E}'_\rho$ that is 0 whenever the above sum for $\mathbb{E}_\rho$ is not well-defined, and is a multiple of $\mathbb{E}_\rho$ when $\mathbb{E}_\rho$ is well-defined.

Note that when $\gamma \in \Gamma_\infty^+$, $1(\tau)|\gamma = 1(\tau)$. Thus taking $\gamma_j^* \in \Gamma(\mathcal{N})$ so that

$$\Gamma_\infty^+\Gamma(\mathcal{N}) = \cup_j \Gamma_\infty^+\gamma_j^* \text{ (disjoint)},$$

and setting

$$\mathbb{E}^*(\tau) = \sum_j 1(\tau)|\gamma_j^*,$$

$\mathbb{E}^*$ is well-defined (and converges absolutely uniformly on compact subsets, so is analytic). With

$$\Gamma_\rho^+ = \{\gamma \in \Gamma_0(\mathcal{N}) : \ \Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho\gamma = \Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho \ \},$$

take $\delta_i \in \Gamma_0(\mathcal{N})$, $\delta'_\ell \in \Gamma_\rho^+$ so that

$$\Gamma_0(\mathcal{N}) = \cup_i \Gamma_\rho^+\delta_i \text{ (disjoint)}, \ \Gamma_\rho^+ = \cup_\ell \Gamma(\mathcal{N})\delta'_\ell \text{ (disjoint)}$$

(note that $\Gamma(\mathcal{N}) \subseteq \Gamma_\rho^+$). Thus

$$\Gamma_0(\mathcal{N}) = \cup_{i,\ell}\Gamma(\mathcal{N})\delta'_\ell\delta_i \text{ (disjoint)}.$$

Set $G_\pm = \begin{pmatrix} I_{n-1} & \\ & -1 \end{pmatrix}$, $\gamma_\pm = \begin{pmatrix} G_\pm & \\ & G_\pm \end{pmatrix}$; remembering $\Gamma(\mathcal{N})$ is a normal subgroup of $Sp_n(\mathbb{Z})$, we have

$$\begin{aligned}\Gamma_\infty\gamma_\rho\Gamma_0(\mathcal{N}) &= \cup_{i,\ell}\left(\Gamma_\infty^+\gamma_\rho\Gamma(\mathcal{N})\delta'_\ell\delta_i \cup \Gamma_\infty^+\gamma_\pm\gamma_\rho\Gamma(\mathcal{N})\delta'_\ell\delta_i\right)\\ &= \cup_{i,\ell}\left(\Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho\delta'_\ell\delta_i \cup \Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\pm\gamma_\rho\delta'_\ell\delta_i\right).\end{aligned}$$

Now set

$$\mathbb{E}'_\rho = \sum_{i,\ell}\overline{\chi}(\delta'_\ell\delta_i)\,\mathbb{E}^*|\gamma_\rho\delta'_\ell\delta_i \ + \sum_{i,\ell}\overline{\chi}(\gamma_\pm\delta'_\ell\delta_i)\,\mathbb{E}^*|\gamma_\pm\gamma_\rho\delta'_\ell\delta_i.$$

Since $\Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\pm = \gamma_\pm\Gamma_\infty^+\Gamma(\mathcal{N})$, we have

$$\mathbb{E}^*|\gamma_\pm = (-1)^k\mathbb{E}^*;$$

hence $\mathbb{E}'_\rho = 0$ if $\chi(-1) \neq (-1)^k$.

Assume now that $\chi(-1) = (-1)^k$. Then, since $\Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho\delta'_\ell = \Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho$, we have $\mathbb{E}^*|\gamma_\rho\delta'_\ell = \mathbb{E}^*|\gamma_\rho$, and hence

$$\mathbb{E}'_\rho = 2\left(\sum_\ell \overline{\chi}(\delta'_\ell)\right)\sum_i \overline{\chi}(\delta_i)\mathbb{E}^*|\gamma_\rho\delta_i.$$

Here δ'_ℓ varies over a set of representatives for the group $\Gamma(\mathcal{N})\backslash\Gamma_\rho^+$ (and we know χ is trivial on $\Gamma(\mathcal{N})$), so unless χ is trivial on Γ_ρ^+, we have $\mathbb{E}'_\rho = 0$.

Note that $\gamma_\pm \in \Gamma(\mathcal{N})$ if and only if $\mathcal{N} \leq 2$. So when $\mathcal{N} \leq 2$, we have $\Gamma_\infty\gamma_j^*$ varying twice over the distinct elements of $\Gamma_\infty\backslash\Gamma_\infty\Gamma(\mathcal{N})$, and

$$\mathbb{E}^* = \mathbb{E}^*|\gamma_\pm = (-1)^k\mathbb{E}^*.$$

Hence when $\mathcal{N} \leq 2$ and k is odd, $\mathbb{E}^* = 0$, and thus $\mathbb{E}'_\rho = 0$. When $\mathcal{N} > 2$ or k is even,

$$\lim_{\tau\mapsto i\infty} \mathbb{E}^*(\tau) = \begin{cases} 2 & \text{if } \mathcal{N} \leq 2, \\ 1 & \text{if } \mathcal{N} > 2, \end{cases}$$

and $\lim_{\tau\mapsto i\infty} \mathbb{E}'_\rho(\tau)|\gamma_\rho^{-1} = 2[\Gamma_0(\mathcal{N}) : \Gamma_\rho^+]\lim_{\tau\mapsto i\infty}\mathbb{E}^*(\tau)$. (see §4 [Freitag, 1996]).

Also, with $\gamma'_j = \gamma_\rho^{-1}\gamma_j^*\gamma_\rho$, we have

$$\Gamma_\infty\gamma_\rho\Gamma_0(\mathcal{N}) = \cup_{i,j}\Gamma_\infty\gamma_j^*\gamma_\rho\delta_i = \cup_{i,j}\Gamma_\infty\gamma_\rho\gamma'_j\delta_i.$$

(The above unions over i, j are disjoint when $\mathcal{N} > 2$.)

Thus we have proved the following.

Proposition 2.1. *Assume* $\chi(1) = (-1)^k$.

(1) For $\gamma_\rho \in Sp_n(\mathbb{Z})$, $\mathbb{E}_\rho$ *is well-defined if and only if* χ *is trivial on* Γ_ρ^+. *When well-defined,* $\mathbb{E}_\rho$ *is a nonzero multiple of* $\mathbb{E}'_\rho$, *and* $\mathbb{E}'_\rho \neq 0$ *when* $\mathcal{N} > 2$ *or* k *is even.*

(2) Suppose $\mathcal{N} \leq 2$ *and* k *is odd. Then* $\mathbb{E}'_\rho = 0$, *so either* $\mathbb{E}_\rho$ *is not well-defined or* $\mathbb{E}_\rho = 0$.

Next we give a description of a convenient choice of representatives corresponding to the Eisenstein series.

Proposition 2.2. *For any* $\gamma \in Sp_n(\mathbb{Z})$, *there exists some* $\gamma_\rho = \begin{pmatrix} I & 0 \\ M_\rho & I \end{pmatrix} \in Sp_n(\mathbb{Z})$ *so that* $\gamma \in \Gamma_\infty\gamma_\rho\Gamma_0(\mathcal{N})$. *When* $\mathcal{N}$ *is square-free, take* $\rho = (\mathcal{N}_0, \ldots, \mathcal{N}_n)$ *to be a (degree n) multiplicative partition of* $\mathcal{N}$, *meaning* $\mathcal{N}_0 \cdots \mathcal{N}_n = \mathcal{N}$. *Take* M_ρ *diagonal so that* $M_\rho \equiv \begin{pmatrix} I_d & \\ & 0 \end{pmatrix}$ *(q) for each prime q dividing* $\mathcal{N}_d$ $(0 \leq d \leq n)$*; then as* ρ *varies,* γ_ρ *varies over a set of representatives for* $\Gamma_\infty\backslash Sp_n(\mathbb{Z})/\Gamma_0(\mathcal{N})$. *Further, when* $\mathcal{N}$ *is square-free and* $\gamma = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z})$, *we have* $\gamma \in \Gamma_\infty\gamma_\rho\Gamma_0(\mathcal{N})$ *if and only if* $\text{rank}_q M = \text{rank}_q M_\rho$ *for each prime* $q|\mathcal{N}$ *(where* $\text{rank}_q M$ *denotes the rank of* M *modulo* q*).*

(When $4\nmid\mathcal{N}$, we give a more detailed description of a set of representatives for $\Gamma_\infty\backslash Sp_n(\mathbb{Z})/\Gamma_0(\mathcal{N})$ in §?.)

Proof. Given $\gamma=\begin{pmatrix}*&*\\M&N\end{pmatrix}\in Sp_n(\mathbb{Z})$, note that we have $\gamma\in\Gamma_\infty\gamma_\rho\Gamma_0(\mathcal{N})$ if and only if $(M_\rho\ I)\in GL_n(\mathbb{Z})(M\ N)\Gamma_0(\mathcal{N})$. We proceed algorithmically to first construct a pair $(M'\ N')\in GL_n(\mathbb{Z})(M\ N)\Gamma_0(\mathcal{N})$ with $N'\equiv I\ (\mathcal{N})$.

Fix a prime q dividing $\mathcal{N}$ with $q^t\parallel\mathcal{N}$. By Lemma ??, we can choose $E_0,G_0\in SL_n(\mathbb{Z})$ so that $E_0,G_0\equiv I\ (\mathcal{N}/q^t)$ and $E_0N\,{}^tG_0^{-1}\equiv\begin{pmatrix}N_1&0\\0&0\end{pmatrix}\ (q^t)$ where N_1 is $d\times d$ and invertible modulo q (so $d=\operatorname{rank}_qN$). We can adjust E_0,G_0 so that $N_1\equiv\begin{pmatrix}a&\\&I\end{pmatrix}\ (q^t)$, some a. Similarly, we can choose $\begin{pmatrix}u&v\\w&x\end{pmatrix}\in SL_2(\mathbb{Z})$ so that $\begin{pmatrix}u&v\\w&x\end{pmatrix}\equiv I\ (\mathcal{N}/q^t)$, $\begin{pmatrix}u&v\\w&x\end{pmatrix}\equiv\begin{pmatrix}a&0\\0&\overline{a}\end{pmatrix}\ (q^t)$. Then

$$\gamma_0=\begin{pmatrix}u&&v&\\&I_{n-1}&&\\w&&x&\\&&&I_{n-1}\end{pmatrix}\in\Gamma_0(\mathcal{N})$$

and $E_0(M\ N)\begin{pmatrix}G_0&\\&{}^tG_0^{-1}\end{pmatrix}\gamma_0\equiv\left(\begin{pmatrix}M_1&M_2\\M_3&M_4\end{pmatrix}\ \begin{pmatrix}I_d&\\&0\end{pmatrix}\right)\ (q^t)$ with M_1 $d\times d$. By symmetry, $M_3\equiv0\ (q^t)$; since $(M,N)=1$, M_4 is invertible modulo q. Thus we can find $E_1',G_1'\in SL_{n-d}(\mathbb{Z})$ so that $E_1',G_1'\equiv I\ (\mathcal{N}/q^t)$,

$$M_4'=E_1'M_4G_1'\equiv\begin{pmatrix}I&\\&a'\end{pmatrix}\ (q^t).$$

Take $E_1=\begin{pmatrix}I_d&\\&E_1'\end{pmatrix}$, $G_1=\begin{pmatrix}I_d&\\&G_1'\end{pmatrix}$, $W=\begin{pmatrix}0_d&&\\&I_{n-d-1}&\\&&\overline{a}'\end{pmatrix}$ where $\overline{a}'a'\equiv1\ (q^t)$; then

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$$\begin{aligned}(C\ D)&=E_1E_0(M\ N)\begin{pmatrix}G_0&\\&{}^tG_0^{-1}\end{pmatrix}\gamma_0\begin{pmatrix}G_1&\\&{}^tG_1^{-1}\end{pmatrix}\begin{pmatrix}I&W\\0&I\end{pmatrix}\\&\equiv\left(\begin{pmatrix}M_1&M_2'\\M_3'&M_4'\end{pmatrix}\ I\right)\ (q^t),\end{aligned}$$

and $(C\ D)\in GL_n(\mathbb{Z})(M\ N)\Gamma_0(\mathcal{N})$ with $(C\ D)\equiv(M\ N)\ (\mathcal{N}/q^t)$ and $D\equiv I\ (q^t)$.

Next, suppose p is another prime dividing $\mathcal{N}$ with $p^r\parallel\mathcal{N}$. Applying the above process to the pair $(C\ D)$, we obtain a pair $(C'\ D')\in GL_n(\mathbb{Z})(M\ N)\Gamma_0(\mathcal{N})$ with $(C'\ D')\equiv(M\ N)\ (\mathcal{N}/(q^tp^r))$ and $D'\equiv I\ (q^tp^r)$. Continuing, we obtain $(M'\ N')\in$

$GL_n(\mathbb{Z})(M\ N)\Gamma_0(\mathcal{N})$ with $N' \equiv I\ (\mathcal{N})$. Thus $(\mathcal{N}M'\ N')$ is a coprime symmetric pair, so there exist K', L' so that $\mathcal{N}|L'$ and $\begin{pmatrix} K' & L' \\ M' & N' \end{pmatrix} \in Sp_n(\mathbb{Z})$; note that we must have $K' \equiv I\ (\mathcal{N})$ since $L' \equiv 0\ (\mathcal{N})$ and $N' \equiv I\ (\mathcal{N})$. Since M' is necessarily symmetric modulo $\mathcal{N}$, we can choose a symmetric matrix M'' so that $M'' \equiv M'\ (\mathcal{N})$; set

$$\delta = \begin{pmatrix} {}^tN' & -{}^tL' \\ -{}^tM' & {}^tK' \end{pmatrix} \begin{pmatrix} I & 0 \\ M'' & I \end{pmatrix}.$$

Then $\delta \in \Gamma(\mathcal{N})$, and $(M''\ I) = (M'\ N')\delta \in GL_n(\mathbb{Z})(M\ N)\Gamma_0(\mathcal{N})$.

Now suppose $\mathcal{N}$ is square-free and M is an integral symmetric matrix. We show that there is some $(M'\ N') \in GL_n(\mathbb{Z})(M\ I)\Gamma_0(\mathcal{N})$ so that $N' \equiv I\ (\mathcal{N})$ and $M' \equiv M_\rho\ (\mathcal{N})$ where M_ρ is diagonal and, for each prime q dividing $\mathcal{N}$, $M_\rho \equiv \begin{pmatrix} I_d & \\ & 0 \end{pmatrix}$ (q) where $d = \operatorname{rank}_q M$. Then the argument of the preceeding paragraph gives us $(M_\rho\ I) \in GL_n(\mathbb{Z})(M\ I)\Gamma_0(\mathcal{N})$. So it suffices now to show that for each prime $q|\mathcal{N}$, there are $E \in SL_n(\mathbb{Z})$, $\gamma \in \Gamma_0(\mathcal{N})$ so that $E, \gamma \equiv I\ (\mathcal{N}/q)$, and $E(M\ I)\gamma \equiv (C\ I)\ (q)$ where $C = \begin{pmatrix} I_d & \\ & 0 \end{pmatrix}$ with $d = \operatorname{rank}_q M$.

If $\operatorname{rank}_q M = 0$ then there is nothing to do. Suppose not; first consider the case that q is odd. By §92 of [O'M], we know there exists $E' \in SL_n(\mathbb{Z}_q)$ so that $E'M\,{}^tE'$ is diagonal with $E'M\,{}^tE' \equiv \begin{pmatrix} M_1 & \\ & 0 \end{pmatrix}$ (q), $M_1 = \begin{pmatrix} a & \\ & I \end{pmatrix}$ with $q \nmid a$. Thus we can find $E \in SL_n(\mathbb{Z})$ so that $E \equiv I\ (\mathcal{N}/q)$, $E \equiv E'\ (q)$. Then

$$E(M\ I) \begin{pmatrix} {}^tE & \\ & E^{-1} \end{pmatrix} = (M'\ I)$$

where $M' \equiv (\,E'M\,{}^tE'\,)\ (q)$. Take $\begin{pmatrix} u & v \\ w & x \end{pmatrix} \in SL_2(\mathbb{Z})$ so that $\begin{pmatrix} u & v \\ w & x \end{pmatrix} \equiv I\ (\mathcal{N}/q)$, $\begin{pmatrix} u & v \\ w & x \end{pmatrix} \equiv \begin{pmatrix} \overline{a} & \overline{a}-1 \\ 0 & a \end{pmatrix}$ (q). Set

$$\gamma = \begin{pmatrix} u & & v & \\ & I_{n-1} & & 0 \\ w & & x & \\ & 0 & & I_{n-1} \end{pmatrix}.$$

Then $\gamma \equiv I\ (\mathcal{N}/q)$ and $(M'\ I)\gamma \equiv (C\ I)\ (q)$ where $C = \begin{pmatrix} I_d & \\ & 0 \end{pmatrix}$.

Now suppose $q = 2$. By Lemma ?? there is some $E \in SL_n(\mathbb{Z})$ so that $E \equiv I\ (\mathcal{N}/q)$ and $EM\,{}^tE \equiv \begin{pmatrix} M_1 & \\ & 0 \end{pmatrix}$ (q), where either $M_1 = I_d$ or $M_1 = A_1$, $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ($d \times d$ where $d = \operatorname{rank}_q M$). In the first case, we are done.

Otherwise, take $A \in SL_n(\mathbb{Z})$ so that $A \equiv I\ (\mathcal{N}/q)$ and $A \equiv \begin{pmatrix} A_1 & \\ & I_{n-d} \end{pmatrix}\ (q)$; set $\gamma = \begin{pmatrix} {}^tEA & {}^tE(A-I) \\ & E^{-1}A \end{pmatrix}$. Thus $\gamma \in \Gamma_0(\mathcal{N})$, $\gamma \equiv I\ (\mathcal{N}/q)$, and $E(M\ I)\gamma \equiv (C\ I)\ (q)$ where $C = \begin{pmatrix} I_d & \\ & 0 \end{pmatrix}$. $\square$

Proposition 2.3. *Suppose $\mathcal{N}$ is square-free, χ is a character modulo $\mathcal{N}$ so that $\chi(-1) = (-1)^k$, and $\rho = (\mathcal{N}_0, \dots, \mathcal{N}_n)$ is a multiplicative partition of $\mathcal{N}$ (as in Proposition 2.2; so $\mathcal{N}_0 \cdots \mathcal{N}_n = \mathcal{N}$). Then $\mathbb{E}_\rho$ is well-defined if and only if $\chi_q^2 = 1$ for all primes $q|\mathcal{N}/(\mathcal{N}_0\mathcal{N}_n)$.*

Proof. Suppose q is a prime dividing $\mathcal{N}_d$ where $0 < d < n$. Fix $\alpha \in \mathbb{F}_q^\times$. By Lemma ??, there exist $G = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, G' = \begin{pmatrix} u' & v' \\ w' & x' \end{pmatrix} \in SL_2(\mathbb{Z})$ so that $G, G' \equiv I\ (\mathcal{N}/q)$,

$$G \equiv \begin{pmatrix} \overline{\alpha} & \overline{\alpha} - \alpha \\ 0 & \alpha \end{pmatrix}\ (q),\ G' \equiv \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}\ (q).$$

Let A, B, C, D, E, W be the $n \times n$ matrices

$$A = \begin{pmatrix} u & & \\ & I & \\ & & u' \end{pmatrix},\ B = \begin{pmatrix} v & & \\ & 0 & \\ & & v' \end{pmatrix},\ C = \begin{pmatrix} w & & \\ & 0 & \\ & & w' \end{pmatrix},$$

$$D = \begin{pmatrix} x & & \\ & I & \\ & & x' \end{pmatrix},\ E = \begin{pmatrix} u' & & v' \\ & I & \\ w' & & x' \end{pmatrix},\ W = \begin{pmatrix} x^2 - 1 & \\ & 0 \end{pmatrix}.$$

Then $\gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(\mathcal{N})$, $E \in SL_n(\mathbb{Z})$, and

$$\delta = \begin{pmatrix} E & \\ & {}^tE^{-1} \end{pmatrix} \begin{pmatrix} I & W \\ & I \end{pmatrix} \in \Gamma_\infty^+.$$

Further, $\delta\gamma_\rho\gamma' \equiv \gamma_\rho^+\ (\mathcal{N})$. Set $\gamma'' = (\delta\gamma_\rho\gamma')^{-1}\gamma_\rho$. So $\gamma'' \in \Gamma(\mathcal{N})$, $\gamma'\gamma'' \in \Gamma_\rho$ with $\chi(\gamma'\gamma'') = \chi_q^2(\alpha)$. Thus the condition that $\chi_q^2 = 1$ for all primes $q|\mathcal{N}/(\mathcal{N}_0\mathcal{N}_n)$ is necessary for $\mathbb{E}_\rho$ to be well-defined.

Now suppose $\chi_q^2 = 1$ for all primes $q|\mathcal{N}/(\mathcal{N}_0\mathcal{N}_n)$, and suppose $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\rho^+$. Thus there exist $\delta = \begin{pmatrix} {}^tE^{-1} & WE \\ & E \end{pmatrix} \in \Gamma_\infty^+$, $\gamma' \in \Gamma(\mathcal{N})$ so that $\delta\gamma'\gamma_\rho\gamma = \gamma_\rho$. Fix a prime $q|\mathcal{N}_d$, $0 \le d \le n$.

When $d = 0$, we have $ED \equiv I\ (q)$, so $\det D \equiv \det \overline{E} \equiv 1\ (q)$ and $\chi_q(\det D) = 1$. When $d = n$, we have $EA \equiv I \equiv A\,{}^tD\ (q)$, so $\det D \equiv \det E \equiv 1\ (q)$ and $\chi_q(\det D) = 1$.

Now suppose $0 < d < n$. Write

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},\ D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},\ E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

where A_1, D_1, E_1 are $d \times d$. Then we have $E_3(A_1\ A_2) \equiv 0\ (q)$; since the rows of $(A_1\ A_2)$ are linearly independent modulo q, we must have $E_3 \equiv 0\ (q)$. Also,

$$E_1(A_1\ A_2) \equiv (I_d\ 0)\ (q),\ E_4(D_3\ D_4) \equiv (0\ I_{n-d})\ (q),$$

so $A_2, D_3 \equiv 0\ (q)$, $A_1 \equiv \overline{E}_1\ (q)$, $D_4 \equiv \overline{E}_4\ (q)$. Since $A\,{}^tD \equiv I\ (q)$, we must have $D_1 \equiv {}^tE_1\ (q)$. Thus we have

$$\det D \equiv \det E_1 \cdot \det \overline{E}_4 \equiv (\det E_1)^2\ (q)$$

and

$$\chi_q(\det D) = \chi_q^2(\det E_1) = 1.$$

Consequently $\chi(\gamma) = \chi(\det D) = 1$, and hence the condition that $\chi_q^2 = 1$ for all primes $q|\mathcal{N}/(\mathcal{N}_0\mathcal{N}_n)$ is sufficient for $\mathbb{E}_\rho$ to be well-defined. $\square$

We now give a robust definition of $\mathbb{E}_\rho$.

Definition. Having fixed $n, k, \mathcal{N} \in \mathbb{Z}_+$ with $k \geq n+2$, χ a character modulo $\mathcal{N}$, and $\gamma_\rho \in Sp_n(\mathbb{Z})$, we define

$$\mathbb{E}_\rho = \begin{cases} \frac{1}{2[\Gamma_0(\mathcal{N}):\Gamma_\rho^+]}\mathbb{E}'_\rho & \text{if } \mathcal{N} > 2, \\ \frac{1}{4[\Gamma_0(\mathcal{N}):\Gamma_\rho^+]}\mathbb{E}'_\rho & \text{if } \mathcal{N} \leq 2. \end{cases}$$

Remark. Suppose that $G_\pm M_\rho = M_\rho G_\pm$. Then for $G \in GL_n(\mathbb{Z})$, $\gamma \in \Gamma_0(\mathcal{N})$, we have $G(M_\rho\ I)\gamma = GG_\pm(M_\rho\ I)\gamma_\pm\gamma$. So with $\gamma_\rho = \begin{pmatrix} I & 0 \\ M_\rho & I \end{pmatrix}$, we have $\Gamma_\infty\Gamma(\mathcal{N})\gamma_\rho\gamma = \Gamma_\infty\Gamma(\mathcal{N})\gamma_\rho\gamma_\pm\gamma$ (since $\gamma_\pm \in \Gamma_\infty$), but $\Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho\gamma = \Gamma_\infty^+\Gamma(\mathcal{N})\gamma_\rho\gamma_\pm\gamma$ if and only if $\mathcal{N} \leq 2$ (since $\gamma_\pm \in \Gamma(\mathcal{N})$ if and only if $\mathcal{N} \leq 2$). Thus,

$$\mathbb{E}_\rho(\tau) = m_\rho \sum_\gamma \overline{\chi}(\gamma)\, 1(\tau)|\gamma_\rho\gamma$$

where γ varies so that $\Gamma_\infty^+\gamma_\rho\Gamma_0(\mathcal{N}) = \cup_\gamma \Gamma_\infty^+\gamma_\rho\gamma$ (disjoint), and

$$m_\rho = \begin{cases} 1 & \text{if } \mathcal{N} \leq 2, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

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We let $\mathcal{E}_k^{(n)}(\mathcal{N}, \chi)$ denote the space spanned by these forms.

§3. Defining Hecke operators

For each prime p, we define Hecke operators $T(p)$, $T_j(p^2)$ $(1 \le j \le n)$ acting on Siegel modular forms; then we describe explicit sets of matrices that give the action of these operators.

Fix a prime p; set $\Gamma = \Gamma_0(\mathcal{N})$ and take $f \in \mathcal{M}_k^{(n)}(\mathcal{N},\chi)$. We define

$$f|T(p) = p^{n(k-n-1)/2} \sum_{\gamma} \overline{\chi}(\gamma)\, f|\delta^{-1}\gamma$$

where $\delta = \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix}$, γ varies over $(\delta\Gamma\delta^{-1} \cap \Gamma)\backslash\Gamma$, and for $\gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$,

$$f(\tau)|\gamma' = (\det \gamma')^{k/2} \det(C\tau + D)^{-k} f((A\tau+B)(C\tau+D)^{-1}).$$

We define

$$f|T_j(p^2) = p^{j(k-n-1)} \sum_{\gamma} \overline{\chi}(\gamma)\, f|\delta_j^{-1}\gamma$$

where $\delta_j = \begin{pmatrix} X_j & \\ & X_j^{-1} \end{pmatrix}$, $X_j = \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix}$, and γ varies over $(\delta_j\Gamma\delta_j^{-1} \cap \Gamma)\backslash\Gamma$.

Proposition 3.1. *Let p be a prime, $f \in \mathcal{M}_k^{(n)}(\mathcal{N},\chi)$. For $0 \le r, n_0+n_2 \le n$, let*

$$D_r = \begin{pmatrix} pI_r & \\ & I \end{pmatrix},\ D_{n_0,n_2} = \begin{pmatrix} pI_{n_0} & & \\ & I & \\ & & \frac{1}{p}I_{n_2} \end{pmatrix}\ (n \times n),$$

and let

$$\mathcal{K}_r = D_r SL_n(\mathbb{Z}) D_r^{-1} \cap SL_n(\mathbb{Z}),$$
$$\mathcal{K}_{n_0,n_2} = D_{n_0,n_2} SL_n(\mathbb{Z}) D_{n_0,n_2}^{-1} \cap SL_n(\mathbb{Z}).$$

Then

$$f|T(p) = p^{n(k-n-1)/2} \sum_{0\le r\le n} \chi(p^{n-r}) \sum_{G,Y} f| \begin{pmatrix} D_r^{-1} & \\ & \frac{1}{p}D_r \end{pmatrix} \begin{pmatrix} G^{-1} & Y\,{}^tG \\ & {}^tG \end{pmatrix}$$

where G varies over $SL_n(\mathbb{Z})/\mathcal{K}_r$ and Y varies over

$$\mathcal{Y}_r = \left\{ \begin{pmatrix} Y_0 & \\ & 0 \end{pmatrix} \in \mathbb{Z}^{n,n}_{\text{sym}} :\ Y_0\ r \times r,\ \textit{varying modulo } p \right\}.$$

Also,

$$f|T_j(p^2)$$
$$= p^{j(k-n-1)} \sum_{n_0+n_2\le j} \chi(p^{j-n_0+n_2}) \sum_{G,Y} f| \begin{pmatrix} D_{n_0,n_2}^{-1} & \\ & D_{n_0,n_2} \end{pmatrix} \begin{pmatrix} G^{-1} & Y\,{}^tG \\ & {}^tG \end{pmatrix}$$

where G varies over $SL_n(\mathbb{Z})/\mathcal{K}_{n_0,n_2}$ and Y varies over $\mathcal{Y}_{n_0,n_2}$, the set of all integral, symmetric $n \times n$ matrices

$$\begin{pmatrix} Y_0 & Y_2 & Y_3 & 0 \\ {}^tY_2 & Y_1/p & 0 & \\ {}^tY_3 & 0 & & \\ 0 & & & \end{pmatrix}$$

with Y_0 $n_0 \times n_0$, varying modulo p^2, Y_1 $(j-n_0-n_2)\times(j-n_0-n_2)$, varying modulo p provided $p \nmid \det Y_1$, Y_2 $n_0 \times (j-n_0-n_2)$, varying modulo p, and Y_3 $n_0 \times (n-j)$, varying modulo p.

Proof. Fix $\Lambda = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$ (a reference lattice).

By Lemma ??, as G varies over $SL_n(\mathbb{Z})/\mathcal{K}_r$, $\Omega = \Lambda G D_r$ varies over all lattices Ω, $p\Lambda \subseteq \Omega \subseteq \Lambda$ with $[\Lambda : \Omega] = p^r$. Thus by Proposition 3.1 and (the proof of) Theorem 6.1 in [HW], claim (1) of the proposition follows.

For Ω another lattice on $\mathbb{Q}\Lambda$, let $\mathrm{mult}_{\{\Lambda:\Omega\}}(a)$ be the multiplicity of the value of a among the invariant factors $\{\Lambda : \Omega\}$. By Lemma ??, as G varies over $SL_n(\mathbb{Z})/\mathcal{K}_{n_0,n_2}$, $\Omega = \Lambda G D_{n_0,n_2}$ varies over all lattices Ω, $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, with $\mathrm{mult}_{\{\Lambda:\Omega\}}(1/p) = n_2$, $\mathrm{mult}_{\{\Lambda:\Omega\}}(p) = n_0$. Thus by Proposition 3.1 and (the proofs of) Theorems 4.1 and 6.1 in [HW], claim (2) of the proposition follows. □

Remark. For $\mathcal{N}' \in \mathbb{Z}_+$ so that $p \nmid \mathcal{N}'$, we can choose G, Y in the above proposition so that $G \equiv I$ $(\mathcal{N}')$ and $Y \equiv 0$ $(\mathcal{N}')$. Also, if $p|\mathcal{N}$, then

$$f|T(p) = p^{n(k-n-1)/2} \sum_Y f| \begin{pmatrix} \frac{1}{p} I_n & \frac{1}{p} Y \\ & I_n \end{pmatrix}$$

where Y varies over $\mathcal{Y}_n$, and

$$f|T_j(p^2) = p^{j(k-n-1)} \sum_{G,Y} f| \begin{pmatrix} D_{j,0}^{-1} & \\ & D_{j,0} \end{pmatrix} \begin{pmatrix} G^{-1} & Y\,{}^tG \\ & {}^tG \end{pmatrix}$$

where G varies over $SL_n(\mathbb{Z})/\mathcal{K}_{j,0}$ and Y varies over $\mathcal{Y}_{j,0}$.

LYNNE: CHECK THESE ABOVE SUMS

§4. Hecke operators on Siegel Eisenstein series of square-free level

Throughout this section, we assume $\mathcal{N}$ is square-free, χ is a character modulo $\mathcal{N}$ so that $\chi(-1) = (-1)^k$; further, we assume either $\mathcal{N} > 2$ or k is even. Take a multiplicative partition $\rho = (\mathcal{N}_0, \dots, \mathcal{N}_n)$ of $\mathcal{N}$ (so $\mathcal{N}_0 \cdots \mathcal{N}_n = \mathcal{N}$), and assume that $\mathbb{E}_\rho \neq 0$ (so by Proposition 2.3, $\chi_{q'}^2 = 1$ for all primes $q'|\mathcal{N}/(\mathcal{N}_0\mathcal{N}_n)$). Take diagonal M_ρ as in Proposition 2.2, $\gamma_\rho = \begin{pmatrix} I & 0 \\ M_\rho & I \end{pmatrix}$.

With $\beta = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in SL_n(\mathbb{Z})$ and $\gamma \in \Gamma_0(\mathcal{N})$ so that $\Gamma_\infty^+ \beta = \Gamma_\infty^+ \gamma_\rho \gamma$, we can determine how to compute $\chi(\gamma)$ from $(M\ N)$.

Suppose $\begin{pmatrix} * & * \\ M & N \end{pmatrix} \in \Gamma_\infty^+ \gamma_\rho \Gamma_0(\mathcal{N})$; so $(M\ N) = E'(M_\rho\ I)\gamma$ for some $E' \in SL_n(\mathbb{Z})$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(\mathcal{N})$. Fix q and take $d = \mathrm{rank}_q M_\rho$. Thus $\mathrm{rank}_q M_\rho = d$, so we can find $E, G \in SL_n(\mathbb{Z})$ so that $EMG \equiv \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}$ (q) where M_1 is $d \times d$ and invertible modulo q. Write $EN\,{}^tG^{-1} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$ where N_1 is $d \times d$; since $M\,{}^tN$ is symmetric, we must have $N_3 \equiv 0$ (q). Hence

$$EMG \equiv \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \equiv EE' \begin{pmatrix} I_d & \\ & 0 \end{pmatrix} AG\ (q),$$

$$\begin{pmatrix} N_1 & N_2 \\ 0 & N_4 \end{pmatrix} \equiv EE' \left(\begin{pmatrix} I_d & \\ & 0 \end{pmatrix} B + D \right) {}^tG^{-1}\ (q).$$

Given the shape of EMG, we must have $EE' \equiv \begin{pmatrix} E_1 & E_2 \\ 0 & E_4 \end{pmatrix}$ (q) where E_1 is $d \times d$ and E_1, E_4 are invertible modulo q, and then $AG \equiv \begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}$ (q) where A_1 is $d \times d$; since $\mathcal{N}|C$, A_1, A_4 are invertible modulo q. We have $A\,{}^tD \equiv I$ (q), so $D\,{}^tG^{-1} \equiv \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}$ (q) where D_1 is $d \times d$ and D_1, D_4 are invertible modulo q. Further, we must have

$$A_1\,{}^tD_1 \equiv I_d,\ A_4\,{}^tD_4 \equiv I_{n-d},\ E_1A_1 \equiv M_1,\ E_4D_4 \equiv N_4\ (q).$$

So

$$\det \overline{M}_1 \cdot \det N_4 \equiv \det \overline{E}_1 \cdot \det E_4 \cdot \det \overline{A}_1 \cdot \det D_4 \equiv (\det \overline{E}_1)^2 \cdot \det D\ (q).$$

Note that when $d = 0$ $D \equiv N$ (q), and when $d = n$, ${}^tD \equiv \overline{A} \equiv \overline{M}$ (q). When $0 < d < n$, we have $\chi_q^2 = 1$ so

$$\chi_q(\det \overline{M}_1 \cdot \det N_4) = \chi_q(\det D).$$

Thus we can define $\chi_q(M,N) = \chi_q(\det \overline{M}_1 \cdot \det N_4)$, and

$$\chi(M,N) = \prod_{q|\mathcal{N}} \chi_q(M,N).$$

Then we have

$$\mathbb{E}_\rho(\tau) = \frac{1}{2} \sum_{(M\ N)} \overline{\chi}(M,N)\ \det(M\tau + N)^{-k}$$

where $(M\ N)$ varies over coprime symmetric pairs so that

$$SL_n(\mathbb{Z})(M_\rho\ I)\Gamma_0(\mathcal{N}) = \cup_{(M\ N)} SL_n(\mathbb{Z})(M\ N)\ \text{(disjoint)}.$$

Now we prove the following.

Theorem 4.1. *Fix a prime $q|\mathcal{N}$, and fix a multiplicative partition $\sigma = (\mathcal{N}'_0, \dots, \mathcal{N}'_n)$ of $\mathcal{N}/q$. For $0 \le d \le n$, let $\mathbb{E}_{\sigma_d}$ denote $\mathbb{E}_{\rho'}$ where $\rho' = (\mathcal{N}_0, \dots, \mathcal{N}_n)$,*

$$\mathcal{N}_i = \begin{cases} \mathcal{N}'_i & \text{if } i \neq d, \\ q\mathcal{N}'_d & \text{if } i = d. \end{cases}$$

Then

$$\mathbb{E}_{\sigma_d}|T(q) = q^{kd-d(d+1)/2}\chi_{\mathcal{N}/q}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q}I_{n-d} \end{pmatrix} M_{\sigma_d}, \begin{pmatrix} qI_d & \\ & I_{n-d} \end{pmatrix}\right)$$
$$\cdot \sum_{t=0}^{n-d} q^{-dt-t(t-1)/2}\beta(d+t,t)\, sym_q^{\chi}(t)\mathbb{E}_{\sigma_{d+t}}$$

where

$$sym_q^{\chi}(t) = \sum_U \chi_q(detU),$$

U varying over $\mathbb{F}^{t,t}_{\mathrm{sym}}$.

Remark. In Lemma ?? we evaluate $\mathrm{sym}_q^{\chi}(t)$.

?? WHAT IF $n-\ell = 0$ and $\chi_1 \neq 0$? Have $\mathbb{E}_t = 0$ for $0 < t < n$. How do we modify this argument to get $\mathbb{E}_0|T(q) = \mathbb{E}_0 + **\mathbb{E}_n$??

Proof.

LYNNE: ?? $n - \ell \mapsto d$??

Write $\mathbb{E}_d$ for $\mathbb{E}_{\sigma_d}$. We know $\mathbb{E}_d(\tau)$ is a sum over representatives for $SL_n(\mathbb{Z})$-equivalence classes of coprime pairs $(M\ N)$ with $\mathrm{rank}_q M = d$; we can assume q divides the lower $n-d$ rows of M. By Proposition 3.1,

$$\begin{aligned} \mathbb{E}_d(\tau)|T(q) &= q^{-n(n+1)/2} \sum_{M,N,Y} \det(M\tau/q + MY/q + N)^{-k} \\ &= q^{kn-n(n+1)/2} \sum_{M,N,Y} \det(M\tau + MY + qN)^{-k} \end{aligned}$$

where Y varies over $\mathcal{Y}_n$. We have

$$\det(M\tau + MY + qN)^{-k} = q^{-k(n-d)} \det(M'\tau + N')^{-k}$$

where

$$(M'\ N') = \begin{pmatrix} I_d & \\ & \frac{1}{q}I_{n-d} \end{pmatrix} (M\ MY + qN).$$

We know the upper d rows of M are linearly independent modulo q, as are the lower $n-d$ rows of N. Thus $(M', N') = 1$, and $\mathrm{rank}_q M' \ge d$. Also note that

$$\det(M\tau + MY + qN)^{-k} = q^{-(n-d)k} \det(M'\tau + N')^{-k}.$$

Recall that we can assume $Y \equiv 0\ (\mathcal{N}/q)$. Also, we know $\mathbb{E}_d$ is supported on the $\Gamma_0(\mathcal{N})$-orbit of $GL_n(\mathbb{Z})(M_\rho\ I)$. Take $(M\ N) = (M_\rho\ I)\gamma$ where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(\mathcal{N})$. Take a prime $q'|\mathcal{N}$ and let $d' = \mathrm{rank}_{q'} M_\rho$. Choose $E \in SL_n(\mathbb{Z})$ so that $AE \equiv \begin{pmatrix} A_1 & 0 \\ * & * \end{pmatrix}$ (q') where A_1 is $d' \times d'$ (possible since we necessarily have $\mathrm{rank}_{q'} A = n$ since $q'|\mathcal{N}$). Then since $A\,{}^tD \equiv I\ (q')$, we have $D\,{}^tE^{-1} \equiv \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}$ (q') with D_1 $d' \times d'$. Thus

$$(M\ N)\begin{pmatrix} E & \\ & {}^tE^{-1} \end{pmatrix} \equiv \begin{pmatrix} A_1 & 0 & * & * \\ 0 & 0 & 0 & D_4 \end{pmatrix}\ (q'),$$

and

$$(M'\ N')\begin{pmatrix} E & \\ & {}^tE^{-1} \end{pmatrix} \equiv \begin{pmatrix} A_1' & 0 & * & * \\ 0 & 0 & 0 & D_4' \end{pmatrix}\ (q')$$

where, modulo q',

$$A_1' \equiv \begin{cases} A_1 & \text{if } d' \le d, \\ \begin{pmatrix} \frac{1}{q} I_d & \\ & I \end{pmatrix} A_1 & \text{if } d' \ge d; \end{cases}$$

$$D_4' \equiv \begin{cases} \begin{pmatrix} qI & \\ & I_{n-d} \end{pmatrix} D_4 & \text{if } d' \le d, \\ D_4 & \text{if } d' \ge d. \end{cases}$$

Therefore

$$\begin{aligned} \chi_{q'}(M', N') &= \chi_{q'}(M'E, N'\,{}^tE^{-1}) = \chi_{q'}(\det \overline{A}_1' \cdot \det D_4') \\ &= \chi_{q'}(q^{d-d'}) \cdot \chi_{q'}(\det \overline{A}_1 \cdot \det D_4), \\ \chi_{q'}(\det \overline{A}_1 \cdot \det D_4) &= \chi_{q'}(M, N), \\ \chi_{q'}(q^{d-d'}) &= \chi_{q'}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q} I_{n-d} \end{pmatrix} M, \begin{pmatrix} qI_d & \\ & I_{n-d} \end{pmatrix} N\right). \end{aligned}$$

Hence

$$\begin{aligned} \chi_{q'}(M', N') &= \chi_{q'}(M'E, N'\,{}^tE^{-1}) \\ &= \chi_{q'}(\det \overline{A}_1' \cdot \det D_4') \\ &= \chi_{q'}\left(\begin{pmatrix} I & \\ & \frac{1}{q} I_{n-d} \end{pmatrix} M_\rho, \begin{pmatrix} qI & \\ & I_{n-d} \end{pmatrix}\right) \chi_{q'}(M, N). \end{aligned}$$

Therefore $\overline{\chi}_{\mathcal{N}/q}(M, N) = \chi_{\mathcal{N}/q}\left(\begin{pmatrix} I & \\ & \frac{1}{q} I_{n-d} \end{pmatrix} M_\rho, \begin{pmatrix} qI & \\ & I_{n-d} \end{pmatrix}\right) \overline{\chi}_{\mathcal{N}/q}(M', N')$.

Reversing, take $(M'\ N')$ a coprime symmetric pair with $\mathrm{rank}_q M' = d+t$; assume $\mathbb{E}_{\sigma,d+t} \neq 0$. We need to count the equivalence classes $SL_n(\mathbb{Z})(M\ N)$ so that

$$\begin{pmatrix} I_d & \\ & \frac{1}{q} I_{n-d} \end{pmatrix} (M\ MY + qN) \in SL_n(\mathbb{Z})(M'\ N').$$

For any $E \in SL_n(\mathbb{Z})$, we have $\begin{pmatrix} I_d & \\ & qI_{n-d} \end{pmatrix} E \begin{pmatrix} I_d & \\ & \frac{1}{q} I_{n-d} \end{pmatrix} \in SL_n(\mathbb{Z})$ if and only if $E \in \mathcal{K}_d$. Thus we need to count the number of $E \in \mathcal{K}_d \backslash SL_n(\mathbb{Z})$ and $Y \in \mathbb{Z}^{n,n}_{\mathrm{sym}}$ (varying modulo q) so that

$$(M\ N) = \begin{pmatrix} I_d & \\ & qI_{n-d} \end{pmatrix} E(M'\ (N' - M'Y)/q)$$

is a coprime pair. We can assume the top $d+t$ rows of M' are linearly independent modulo q, and that q divides the lower $n-d-t$ rows of M'. To have $\mathrm{rank}_q M = d$, we need to choose E so that the top d rows of EM' are linearly independent modulo q; using Lemma ?? there are

$$q^{d(n-d-t)}\beta(d+t,d) = q^{d)(n-d-t)}\beta(d+t,t)$$

choices for E. We need to choose Y so that N is integral and $(M,N) = 1$; equivalently, for any $G \in SL_n(\mathbb{Z})$, we need $N\,{}^tG^{-1}$ integral and $(MG, N\,{}^tG^{-1}) = 1$. Using left multiplication by $\mathcal{K}_d$, we can adjust the choice of E so that the lower $n-d-t$ rows of EM' are divisible by q, and then we can choose $G \in SL_n(\mathbb{Z})$ so that

$$EM'G \equiv \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ (q)$$

where M_1 is $d \times d$, M_5 is $t \times t$, and M_1, M_5 are invertible modulo q. Write

$$EN'\,{}^tG^{-1} = \begin{pmatrix} N_1 & N_2 & N_3 \\ N_4 & N_5 & N_6 \\ N_7 & N_8 & N_9 \end{pmatrix},\ G^{-1}Y\,{}^tG^{-1} = \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ {}^tY_2 & Y_4 & Y_5 \\ {}^tY_3 & {}^tY_5 & Y_6 \end{pmatrix}$$

where N_1, Y_1 are $d \times d$ and N_5, Y_4 are $t \times t$. By symmetry, $N_7, N_8 \equiv 0\ (q)$, and then since $(M', N') = 1$, we must have $\mathrm{rank}_q N_9 = n-d-t$. Also, as Y varies over $\mathbb{F}^{n,n}_{\mathrm{sym}}$, so does $G^{-1}Y\,{}^tG^{-1}$. To have N integral, we need $(Y_1\ Y_2\ Y_3) \equiv \overline{M}_1(N_1\ N_2\ N_3)\ (q)$. Then by symmetry, we find $N_4 \equiv M_5\,{}^tY_2\ (q)$. So to have $(M,N) = 1$, we need $\mathrm{rank}_q(N_5 - M_5Y_4) = t$, or equivalently,

$$\mathrm{rank}_q(N_5 - M_5Y_4)\,{}^tM_5 = t.$$

As Y_4 varies over $\mathbb{F}^{t,t}_{\mathrm{sym}}$, so does $N_5 - M_5Y_4\,{}^tM_5$. We have

$$\begin{aligned}
\chi_q(M,N) &= \chi_q(\det \overline{M}_1 \cdot \det(N_5 - Y_4M_5) \cdot \det N_9) \\
&= \chi_q(\det \overline{M}_1 \cdot \det \overline{M}_5 \det N_9) \cdot \chi_q(\det(N_5 - M_5Y_4)\,{}^tM_5) \\
&= \chi_q(M', N') \cdot \chi_q(\det(N_5 - M_5Y_4)\,{}^tM_5).
\end{aligned}$$

We have no constraints on Y_5 and Y_6, so as we vary Y subject to the above conditions, we get

$$\begin{aligned}\sum_Y \overline{\chi}_q(M,N) &= \overline{\chi}_q(M',N')\cdot q^{(n-d-t)(n-d+t+1)/2}\sum_{U\in\mathbb{F}^{t,t}_{\mathrm{sym}}}\overline{\chi}_q(\det U)\\ &= \overline{\chi}_q(M',N')q^{(n-d-t)(n-d+t+1)/2}\mathrm{sym}^{\chi}_q(t),\end{aligned}$$

as claimed. □

This theorem allows us to diagonalise the space of Eisenstein series. To aid in our description of this, we define a partial ordering on multiplicative partitions of $\mathcal{N}$, as follows.

Definition. For ρ,β multiplicative partitions of $\mathcal{N}$ and $Q|\mathcal{N}$, we write $\beta=\rho\ (Q)$ if, for every prime $q|Q$, we have $\mathrm{rank}_q M_\beta=\mathrm{rank}_q M_\rho$. Similarly, we write $\beta>\rho\ (Q)$ if, for every prime $q|Q$, we have $\mathrm{rank}_q M_\beta>\mathrm{rank}_q M_\rho$.

Corollary 4.2. *Let q be a prime dividing $\mathcal{N}$. For ρ a partition of $\mathcal{N}$ so that $\mathbb{E}_\rho\neq 0$, there are $a_{\rho,\alpha}(q)\in\mathbb{C}$ so that $a_{\rho,\rho}(q)=1$ and*

$$\sum_{\substack{\alpha=\rho\ (\mathcal{N}/q)\\ \alpha\geq\rho\ (q)}} a_{\rho,\alpha}(q)\,\mathbb{E}_\alpha$$

is an eigenform for $T(q)$ with eigenvalue

$$\lambda_\rho(q)=q^{kd-d(d+1)/2}\chi_{\mathcal{N}/q}\left(\begin{pmatrix}I_d&\\&\frac{1}{q}I\end{pmatrix}M_\rho,\begin{pmatrix}qI_d&\\&I\end{pmatrix}\right)$$

where $d=\mathrm{rank}_q M_\rho$. Further, suppose $\alpha=\rho(\mathcal{N}/q)$, $\alpha>\rho\ (q)$, with $d=\mathrm{rank}_q M_\rho$, $d+t=\mathrm{rank}_q M_\alpha$; then we have $a_{\rho,\alpha}(q)\neq 0$ if and only if either (1) $\chi_q=1$, or (2) $\chi_q^2=1$ and t is even.

Proof. By Lemma ?? $\mathrm{sym}^{\chi}_q(t)=0$ if and only if (1) $\chi_q^2\neq 1$, or (2) $\chi_q\neq 1$ and t is odd. Thus by Theorem 4.1,

$$\begin{aligned}\mathrm{span}\Big\{\mathbb{E}_\alpha:\ &\alpha=\rho\ (\mathcal{N}/q),\ \alpha\geq\rho\ (q),\ \text{so that either (1) }\chi_q=1,\text{ or}\\ &\text{(2) }\chi_q^2=1\text{ and }\mathrm{rank}_q M_\alpha-\mathrm{rank}_q M_\rho\text{ is even}\Big\}\end{aligned}$$

is invariant under $T(q)$, and the matrix for $T(q)$ on this basis is upper triangular with nonzero upper triangular entries. Then the standard process of diagonalising an upper triangular matrix yields the result. □

We now prove a multiplicity-one result for the Eisenstein series of square-free level.

Corollary 4.3. *Suppose $\mathbb{E}_\rho \neq 0$. For $\alpha \geq \rho\ (Q)$ and prime $q|Q$, set $a_{\rho,\alpha}(q) = a_{\rho,\sigma}(q)$ where $\sigma = \rho\ (\mathcal{N}/q)$, $\sigma = \alpha\ (q)$, and set*

$$a_{\rho,\alpha}(Q) = \prod_{q|Q} a_{\rho,\alpha}(q).$$

Then with

$$\widetilde{\mathbb{E}}_\rho = \sum_{\alpha \geq \rho\,(\mathcal{N})} a_{\rho,\alpha}(\mathcal{N})\,\mathbb{E}_\alpha,$$

for every prime $q|\mathcal{N}$ we have

$$\widetilde{\mathbb{E}}_\rho | T(q) = \lambda_\rho(q)\,\widetilde{\mathbb{E}}_\rho$$

(where $\lambda_\rho(q)$ is defined in Corollary 4.2).

Proof. Fix a prime $q|\mathcal{N}$. For $\alpha \geq \rho\ (\mathcal{N})$, take $\beta = \alpha\ (\mathcal{N}/q)$, $\beta = \rho\ (q)$. Then $a_{\rho,\alpha}(\mathcal{N}) = a_{\rho,\beta}(\mathcal{N}/q)\,a_{\rho,\alpha}(q)$. Hence

$$\widetilde{\mathbb{E}}_\rho = \sum_{\substack{\beta \geq \rho\ (\mathcal{N}/q)\\ \beta = \rho\ (q)}} a_{\rho,\beta}(\mathcal{N}/q) \sum_{\substack{\alpha = \beta\ (\mathcal{N}/q)\\ \alpha \geq \beta\ (q)}} a_{\rho,\alpha}(q)\,\mathbb{E}_\alpha.$$

We argue that when $a_{\rho,\beta}(\mathcal{N}/q) \neq 0$, we have $a_{\rho,\alpha}(q) = a_{\beta,\alpha}(q)$ and $\lambda_\rho(q) = \lambda_\beta(q)$.

Fix β so that $\beta \geq \rho\ (\mathcal{N}/q)$, $\beta = \rho\ (q)$, and suppose $a_{\rho,\beta}(\mathcal{N}/q) \neq 0$. Take $Q|\mathcal{N}/q$ so that $\beta = \rho\ (\mathcal{N}/Q)$, $\beta > \rho\ (Q)$. Thus $a_{\rho,\beta}(\mathcal{N}/q) = a_{\rho,\beta}(Q)$. Since $a_{\rho,\beta}(Q) \neq 0$, for each prime $q'|Q$ we have either (1) $\chi_{q'} = 1$, or (2) $\chi_{q'}^2 = 1$ and $\operatorname{rank}_{q'} M_\beta - \operatorname{rank}_{q'} M_\rho$ is even.

Suppose q' is a prime dividing Q so that $\chi_{q'} \neq 1$. Set $r = \operatorname{rank}_{q'} M_\rho$, $r + t = \operatorname{rank}_{q'} M_\beta$ (so t is even). Then for $0 \leq d \leq n$,

$$\begin{aligned}
\chi_{q'}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q'}I \end{pmatrix} M_\rho, \begin{pmatrix} qI_d & \\ & I \end{pmatrix}\right) &= \chi_{q'}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q'}I \end{pmatrix}\begin{pmatrix} I_r & \\ & 0 \end{pmatrix}, \begin{pmatrix} qI_d & \\ & I \end{pmatrix}\right)\\
&= \begin{cases} \chi_{q'}(q^{r-d}) & \text{if } d \leq r,\\ \chi_{q'}(q^{d-r}) & \text{if } d \geq r \end{cases}\\
&= \chi_{q'}(q^{d-r})
\end{aligned}$$

(since $\chi_{q'}^2$). Similarly,

$$\chi_{q'}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q'}I \end{pmatrix} M_\beta, \begin{pmatrix} qI_d & \\ & I \end{pmatrix}\right) = \chi_{q'}(q^{d-r-t})$$

and $\chi_{q'}(q^{d-r-t}) = \chi_{q'}(q^{d-r})$ since t is even and $\chi_{q'}^2 = 1$.

For each prime $q''|\mathcal{N}/Q$, we either have $\beta = \rho\ (q'')$ or $\chi_{q''} = 1$. Thus for $0 \leq d \leq n$,

$$\chi_{\mathcal{N}/q}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q'}I \end{pmatrix} M_\rho, \begin{pmatrix} qI_d & \\ & I \end{pmatrix}\right) = \chi_{\mathcal{N}/q}\left(\begin{pmatrix} I_d & \\ & \frac{1}{q'}I \end{pmatrix} M_\beta, \begin{pmatrix} qI_d & \\ & I \end{pmatrix}\right).$$

Hence $\lambda_\beta(q)=\lambda_\rho(q)$. Further, with σ_d, α_d partitions of $\mathcal{N}$ so that $\sigma_d=\rho\ (\mathcal{N}/q)$, $\mathrm{rank}_q M\sigma_d=d$, $\alpha_d=\beta\ (\mathcal{N}/q)$, $\mathrm{rank}_q M_{\alpha_d}=d$, the matrix for $T(q)$ on ${}^t(\mathbb{E}_{\sigma_0},\dots,\mathbb{E}_{\sigma_n})$ is equal to the matrix for $T(q)$ on ${}^t(\mathbb{E}_{\alpha_0},\dots,\mathbb{E}_{\alpha_n})$, and hence $a_{\rho,\sigma_d}(q)=a_{\beta,\alpha_d}(q)$, $0\le d\le n$. $\square$

Now we evaluate the action of $T_j(q^2)$ on $\mathbb{E}_\rho$. Note that since the Hecke operators commute, the multiplicity-one result of Corollary 4.3 tells us that each $\widetilde{\mathbb{E}}_\rho$ is also an eigenform for $T_j(q^2)$. So we could simply do enough computation to find the eigenvalue $\lambda_{\rho,j}(q^2)$, but we take just a bit more effort and give a complete description of $\mathbb{E}_\rho|T_j(q^2)$. Then in Corollary 4.5 we compute the $T_j(q^2)$ eigenvalues.

Theorem 4.4. *Assume $\mathcal{N}$ is square-free, a fix a prime $q|\mathcal{N}$. For σ a multiplicative partition of $\mathcal{N}/q$ and $0\le d\le n$, let $\mathbb{E}_{\sigma_d}$ be the level $\mathcal{N}$ Eisenstein series as in Theorem 4.1; suppose $\mathbb{E}_{\sigma_d}\ne 0$.*

For $0\le j,d\le n$,

$$\mathbb{E}_{\sigma_d}|T_j(q^2)=\sum_{t=0}^{n-d}A_j(d,t)\mathbb{E}_{\sigma_{d+t}};$$

when $\chi_q=1$,

$$\begin{aligned}A_j(d,t)&=q^{(j-t)d-t(t+1)/2}\beta(d+t,t)\\&\cdot\sum_{d_1=0}^{j}\sum_{d_5=0}^{j-d_1}\sum_{d_8=0}^{d_5}q^{a_j(d;d_1,d_5,d_8)}\chi_{\mathcal{N}/q}(D_{d_1,r}M_{\sigma_d}D_j^{-1},D_{d_1,r},D_j)\\&\cdot\beta(d,d_1)\beta(t,d_5)\beta(n-d-t,d_1+n-d-j-d_8)\\&\cdot\beta(t-d_5,d_8)sym_q^\chi(t-d_5-d_8)sym_q^\chi(d_5,d_8),\end{aligned}$$

where $r=j-d_1-d_5+d+8$, and

$$\begin{aligned}&a_j(d;d_1,d_5,d_8)\\&\quad=(k-d)(2d_1+d_5-d_8)+d_1(d_1-d_8-j-1)-d_8(d_5+t)-d_5(d_5+1)/2+d_8(d_8+1)/2.\end{aligned}$$

[LYNNE: DEFINE $sym_q^\chi(b,c)$]

(Note that $sym_q^\chi(t-d_5-d_8)$, $sym_q^\chi(d_5,d_8)$ are evaluated in Lemmas ???.)

Proof. Fix $d=\mathrm{rank}_q M_\rho$; to ease some notation later, set $\ell=n-d$.

$$\mathbb{E}_{n-\ell}|T_j(q^2)=q^{j(k-n-1)}\sum_{G,Y}\mathbb{E}_{n-\ell}|\begin{pmatrix}D_j^{-1}&\\&D_j\end{pmatrix}\begin{pmatrix}G^{-1}&Y\,{}^tG\\&{}^tG\end{pmatrix}$$

where $D_j=\begin{pmatrix}qI_j&\\&I_{n-j}\end{pmatrix}$, $G\in SL_n(\mathbb{Z})/\mathcal{K}_j$, $Y\in\mathcal{Y}_j$ with $\mathcal{Y}_j$ the set of matrices

$\begin{pmatrix} U & V \\ {}^tV & 0 \end{pmatrix}$ so that $U \in \mathbb{Z}^{j,j}_{\text{sym}}$ varies modulo q^2, $V \in \mathbb{Z}^{j,n-j}$ varies modulo q. So

$$\begin{aligned}
&\mathbb{E}_{n-\ell}(\tau)|T_j(q^2) \\
&\quad = q^{j(-n-1)} \sum_{G,Y} \sum_{M,N} \det \left(M \left(D_j^{-1}G^{-1}\tau + D_j^{-1}Y\,{}^tG\right) {}^tG^{-1}D_j^{-1} + N\right)^{-k} \\
&\quad = q^{j(k-n-1)} \sum_{G,Y} \sum_{M,N} \det \left(MD_j^{-1}G^{-1}\tau + MD_j^{-1}Y\,{}^tG + N\,{}^tGD_j\right)^{-k}
\end{aligned}$$

(where $(M\ N)$ varies over coprime symmetric pairs with $\operatorname{rank}_q M = n - \ell$).

Take a coprime symmetric pair $(M\ N)$ with $\operatorname{rank}_q M = n - \ell$. Let d_1 be the rank of the first j columns of M; using row operations, we can assume $M = \begin{pmatrix} M_1 & M_2 \\ qM_3 & M_4 \\ qM_5' & qM_6' \end{pmatrix}$ where M_1 is $d_1 \times j$ (so $\operatorname{rank}_q M_1 = d_1$), M_4 is $d_4 \times (n-j)$ with $\operatorname{rank}_q M_4 = d_4 = n - \ell - d_1$. Correspondingly, write $N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \\ N_5' & N_6' \end{pmatrix}$ where N_1 is $d_1 \times j$ and N_4 is $d_4 \times (n-j)$. Take r so that $\operatorname{rank}_q \begin{pmatrix} M_1 & 0 \\ M_5' & N_5' \end{pmatrix} = n - d_4 - r$; so using row operations, we can assume

$$(qM_5'\ qM_6'\ N_5'\ N_6') = \begin{pmatrix} qM_5 & qM_6 & N_5 & N_6 \\ q^2M_7 & qM_8 & N_7 & qN_8 \end{pmatrix}$$

where M_6, N_6 are $(\ell-r)\times(n-j)$ and $\operatorname{rank}_q \begin{pmatrix} M_1 & 0 \\ M_5 & N_6 \end{pmatrix} = n-d_4-r$. Note that since $(M,N) = 1$, we must have $\operatorname{rank}_q N_7 = r$. Then with $D_{d_1,r} = \begin{pmatrix} qI_{d_1} & & \\ & I & \\ & & \frac{1}{q}I_r \end{pmatrix}$,

$$D_{d_1,r}(M\ N) \begin{pmatrix} D_j^{-1} & \\ & D_j \end{pmatrix} = \begin{pmatrix} M_1 & qM_2 & q^2N_1 & qN_2 \\ M_3 & M_4 & qN_3 & N_4 \\ M_5 & qM_6 & qN_5 & N_6 \\ M_7 & M_8 & N_7 & N_8 \end{pmatrix}$$

has q-rank n. Hence for any $Y \in \mathcal{Y}_j$,

$$(M'\ N') = D_{d_1,r}(M\ N) \begin{pmatrix} D_j^{-1} & \\ & D_j \end{pmatrix} \begin{pmatrix} G^{-1} & Y\,{}^tG \\ 0 & {}^tG \end{pmatrix}$$

is a coprime symmetric pair with $\operatorname{rank}_q M' = n - \ell + t$ for some $t \geq 0$. Note that $\det(M'\tau + N')^{-k} = q^{k(d_1-r)} \det(MD_j^{-1}G^{-1}\tau + MD_j^{-1}Y\,{}^tG + ND_j\,{}^tG)^{-k}$.

Similar to the computation in the proof of Theorem 4.1, we have

$$\chi_{\mathcal{N}/q}(M,N) = \chi_{\mathcal{N}/q}(D_{d_1,r}M_{\sigma_d}D_j^{-1}, D_{d_1,r}D_j)\chi_{\mathcal{N}/q}(M',N').$$

Reversing, take a coprime pair $(M'\ N')$ with $\mathrm{rank}_q M' = n-\ell+t$. We need to count the equivalence classes $SL_n(\mathbb{Z})(M\ N)$ so that

$$D_{d_1,r}(M\ N)\begin{pmatrix} D_j^{-1} & \\ & D_j \end{pmatrix}\begin{pmatrix} G^{-1} & Y\,{}^tG \\ 0 & {}^tG \end{pmatrix} \in SL_n(\mathbb{Z})(M'\ N').$$

For $E_1, E_2 \in SL_n(\mathbb{Z})$ and

$$(M_i\ N_i) = D_{d_1,r}^{-1}E_i(M'\ N')\begin{pmatrix} G & -GY \\ 0 & {}^tG^{-1} \end{pmatrix}\begin{pmatrix} D_j & \\ & D_j^{-1} \end{pmatrix},$$

we have $(M_1\ N_1) \in SL_n(\mathbb{Z})(M_2\ N_2)$ if and only if $E_1 \in \mathcal{K}_{d_1,r}E_2$. Thus we need to count the number of triples E, G, Y with $E \in \mathcal{K}_{d_1,r}\backslash SL_n(\mathbb{Z})$, $G \in SL_n(\mathbb{Z})/\mathcal{K}_j$, $Y \in \mathcal{Y}_j$ so that

$$(M\ N) = D_{d_1,r}^{-1}E(M'\ N')\begin{pmatrix} G & -GY \\ 0 & {}^tG^{-1} \end{pmatrix}\begin{pmatrix} D_j & \\ & D_j^{-1} \end{pmatrix}$$

is an integral coprime pair with $\mathrm{rank}_q M = n-\ell$ (that $M\,{}^tN$ is symmetric is automatic).

For $E, G \in SL_n(\mathbb{Z})$, let $(M_1\ M_2)$ be the top d_1 rows of $EM'G$ with M_1 size $d_1 \times j$; similarly, let $(N_1\ N_2)$ be the top d_1 rows of $EN'\,{}^tG^{-1}$ with N_1 size $d_1 \times j$. To have M integral we need $q|M_2$. To have N integral, we will need to solve

$$N_1 \equiv M_1U + M_2\,{}^tV\ (q^2),\ \ N_2 \equiv M_1V\ (q)$$

Since $(M', N') = 1$ and $q|M_2$, we must have $\mathrm{rank}_q(M_1\ N_1\ N_2) = d_1$; thus we can only solve the above congruences if $\mathrm{rank}_q M_1 = d_1$. So suppose we have chosen E, G to meet this condition; write

$$EM'G = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \\ M_5 & M_6 \\ M_7 & M_8 \end{pmatrix},\ \ EN'\,{}^tG^{-1} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \\ N_5 & N_6 \\ N_7 & N_8 \end{pmatrix}$$

where M_1, N_1 are $d_1 \times j$, M_4, N_4 are $d_4 \times (n-j)$, M_5, N_5 are $(n-r-d)\times j$ where $Y = \begin{pmatrix} U & V \\ {}^tV & 0 \end{pmatrix}\mathcal{Y}_j$. To have $\mathrm{rank}_q M = n-\ell$, we need to have $\mathrm{rank}_q \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \\ 0 & M_6 \end{pmatrix} = n-\ell$; so suppose we have chosen E, G to meet this condition as well. Then, using left multiplication from $\mathcal{K}_{d_1,r}$ and right multiplication from $\mathcal{K}_j$, we can assume $\mathrm{rank}_q M_4 = d_4 = n-\ell-d_1$ and $M_6 \equiv 0\ (q)$. Now write $M_i = (A_i'\ A_i)$, $N_i = (B_i'\ B_i)$ where, for i odd, A_i', B_i' have d_1 columns, and for i even, A_i', B_i' have d_4 columns. By adjusting further using $\mathcal{K}_{d_1,r}$ and $\mathcal{K}_j$, we can assume that $\mathrm{rank}_q A_1' = d_1$, $\mathrm{rank}_q A_4' = d_4$, $A_i' \equiv 0\ (q^2)$ for $i \ne 1, 4$, $A_1, A_3 \equiv 0\ (q)$, and with $d_i = \mathrm{rank}_q A_i$ for $i = 5, 7, 8$, we can assume

$$A_5 \equiv \begin{pmatrix} \alpha_5 & 0 & 0 \\ 0 & 0 & q\alpha_5' \end{pmatrix}\ (q^2),\ \ A_6 \equiv \begin{pmatrix} 0 & 0 \\ q\alpha_6' & 0 \end{pmatrix}\ (q^2),$$

$$A_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ (q), \ A_8 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \alpha_8 \end{pmatrix} \ (q)$$

where α_i is $d_i \times d_i$ (and hence invertibe modulo q), α_5' is $(\ell-r-d_5)\times(j-d_1-d_5-d_7)$, and α_6' is $(\ell - r - d_5) \times (n - j - d_4 - d_8)$; here the first d_5 and last $j - d_1 - d_5 - d_7$ columns of A_7 are 0 modulo q, and the top $r - d_7 - d_8$ and bottom d_8 rows of A_7 are 0 modulo q. Correspondingly, write

$$B_5 = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}, \ B_6 = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix},$$

$$B_7 = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 \\ \delta_4 & \delta_5 & \delta_6 \\ \delta_7 & \delta_8 & \delta_9 \end{pmatrix}, \ B_8 = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \\ \epsilon_5 & \epsilon_6 \end{pmatrix}.$$

Then by symmetry, we have $\beta_4, \beta_5, \gamma_4, \delta_1, \delta_2, \epsilon_2 \equiv 0\ (q)$, and q must divide the lower $\ell - r - d_5$ rows of B_5' and the upper $r - d_7 - d_8$ rows of B_7'.

With $Y = \begin{pmatrix} U & V \\ {}^tV & 0 \end{pmatrix}$ (as above), write

$$U = \begin{pmatrix} U_1 & U_2 \\ {}^tU_2 & U_3 \end{pmatrix}, \ V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$$

where U_1 is $d_1 \times d_1$ and V_1 is $d_1 \times d_4$. To have N integral, we need

$$N_1 \equiv A_1'(U_1\ U_2)\ (q^2), \ N_2 \equiv A_1'(V_1\ V_2)\ (q), \ B_2 \equiv A_4'\,{}^tV_3\ (q).$$

With these (unique) choices of U_1, U_2, V_1, V_2, V_3, the symmetry of $M'\,{}^tN'$ implies that

$$B_3'\,{}^tA_1' \equiv A_4'\,{}^tB_2' \equiv A_4'\,{}^tV_2\,{}^tA_1'\ (q),$$

so we automatically get $B_3' \equiv A_4'\,{}^tV_2\ (q)$. Hence with these choices of U_1, U_2, V_1, V_2, V_3, the top $n-\ell$ rows of N are integral. We have already ensured the top $n-\ell$ rows of M are integral with q-rank $n-\ell$, and we know the lower ℓ rows of M are 0 modulo q. So we need to choose U_3, V_4 so that the lower ℓ rows of N are integral with q-rank ℓ.

By symmetry, we have

$$B_5'\,{}^tA_1' \equiv A_5\,{}^tB_1 + A_6\,{}^tB_2 \equiv A_5\,{}^tU_2\,{}^tA_1' + A_6\,{}^tV_2\,{}^tA_1'\ (q^2),$$

$$B_6'\,{}^tA_4' \equiv A_5\,{}^tB_3 \equiv A_5V_3\,{}^tA_4'\ (q),$$

$$B_7'\,{}^tA_1' \equiv A_7\,{}^tB_1 + A_8\,{}^tB_2 \equiv A_7\,{}^tU_2\,{}^tA_1' + A_8\,{}^tV_2\,{}^tA_1'\ (q).$$

So to have N integral, we need to choose E, G so that $\beta_6 \equiv 0\ (q)$, and U_3 so that $B_5 \equiv A_5U_3\ (q)$. With such choices, the lower ℓ rows of N are congruent modulo q to

$$\begin{pmatrix} 0 & (B_5 - A_5U_3 - A_6\,{}^tV_4)/q & 0 & B_6 - A_5V_4 \\ 0 & B_7 - A_7U_3 - A_8\,{}^tV_4 & 0 & 0 \end{pmatrix}.$$

Also, since $(M',N')=1$, when $\beta_6\equiv 0\ (q)$, we will necessarily have $\mathrm{rank}_q\gamma_3=\ell-r-d_5$ (recall that $\beta_4,\beta_5,\gamma_4\equiv 0\ (q)$). Write

$$U_3=\begin{pmatrix}\mu_1&\mu_2&\mu_3\\ {}^t\mu_2&\mu_4&\mu_5\\ {}^t\mu_3&{}^t\mu_5&\mu_6\end{pmatrix},\ V_4=\begin{pmatrix}\nu_1&\nu_2\\ \nu_3&\nu_4\\ \nu_5&\nu_6\end{pmatrix}$$

where μ_1 is $d_5\times d_5$, μ_4 is $d_7\times d_7$, ν_2 is $d_5\times d_8$, and ν_4 is $d_7\times d_8$. Note that

$$B_7-A_7U_3-A_8\,{}^tV_4\equiv\begin{pmatrix}0&0&\delta_3\\ \delta_4-\alpha_7\,{}^t\mu_2&\delta_5-\alpha_7\mu_4&\delta_6-\alpha_7\mu_5\\ \delta_7-\alpha_8\,{}^t\nu_2&\delta_8-\alpha_8\,{}^t\nu_4&\delta_9-\alpha_8\,{}^t\nu_6\end{pmatrix}\ (q).$$

So to have

$$\mathrm{rank}_q\begin{pmatrix}0&(B_5-A_5U_3-A_6\,{}^tV_4)/q&0&B_6-A_5V_4\\ 0&B_7-A_7U_3-A_8\,{}^tV_4&0&0\end{pmatrix},$$

we need to choose E,G so that $\mathrm{rank}_q\delta_3=r-d_7-d_8$. We know that γ_3 is $(\ell-r-d_5)\times(n-j-d_4-d_8)$ and δ_3 is $(r-d_7-d_8)\times(j-d_1-d_5-d_7)$. Thus if $\beta_6\equiv 0\ (q)$ and $\mathrm{rank}_q\delta_3=r-d_7-d_8$, we have

$$\ell-r-d_5\le n-j-d_4-d_8,\ r-d_7-d_8\le j-d_1-d_5-d_7,$$

and consequently $r=j-d_1-d_5+d_8$ (recall that $n-\ell=d_1+d_4$). Then we use right multiplication from $\mathcal{K}_j$ to modify G so that we can assume $\beta_4\equiv 0\ (q^2)$.

Thus we need to choose $\mathcal{K}_{d_1,r}E$, $G\mathcal{K}_j$ so that (adjusting the coset representatives E,G), the top d_1 rows of EM' have q-rank d_1, the top $d_1+d_4+d_5$ rows of EM' have q-rank $d_1+d_4+d_5$ (where $0\le d_5\le j-d_1$), and q divides rows $d_1+d_4+d_5+1$ through $n-d_7-d_8$ of EM'; Lemma ? tells us that the number of such $\mathcal{K}_{d_1,r}E$ is

$$\beta(d',d+d_5)\beta(n-d',n-r-d-d_5)\beta(d+d_5,d_1)\\ \cdot q^{(d+d_5)(r+d+d_5-d')+d_1(n-d-d_5)}$$

where $d=\mathrm{rank}_qM$, $d'=\mathrm{rank}_qM'$ (note that after choosing E as in the lemma, we can use left multiplication from $\mathcal{K}_{d_1,r}$ to ensure rows $d_1+d_4+d_5+1$ through $n-d_7-d_8$ are divisible by q). Then we can choose some $G_0\in SL_n(\mathbb{Z})$ so that

$$EM'G_0\equiv\begin{pmatrix}C&0&0&0\\ 0&C'&0&0\\ 0&0&0&0\\ 0&0&C''&0\end{pmatrix}\ (q)$$

where C is $d_1\times d_1$ with $\mathrm{rank}_qC=d_1$, C' is $(d_4+d_5)\times(d_4+d_5)$ with $\mathrm{rank}_qC'=d_4+d_5$. As $G\mathcal{K}_j$ varies over $SL_n(\mathbb{Z})/\mathcal{K}_j$, so does $G_0G\mathcal{K}_j$; Lemma ? tells us that the number of $G\mathcal{K}_j$ that meet all the necessary criteria as described above is

$$\beta(d_4+d_5,d_4)\beta(d_7+d_8,d_8)q^{(d_4+d_8)(j-d_1-d_5)-d_7d_8}.$$

Having chosen such E, G, we have seen that to have N integral, U_1, U_2, V_1, V_2, V_3 are uniquely determined, and μ_1, μ_2, μ_3 are determined modulo q. To also have $(M, N) = 1$, we need to ensure $\text{rank}_q B = \ell$ where

$$B = \begin{pmatrix} (\beta_1 - \alpha_5\mu_1)/q & (\beta_2 - \alpha_5\mu_2)/q & (\beta_3 - \alpha_5\mu_3)/q & \gamma_1 - \alpha_5\nu_1 & \gamma_2 - \alpha_5\nu_2 \\ 0 & * & * & \gamma_3 & 0 \\ 0 & 0 & \delta_3 & 0 & 0 \\ 0 & \delta_5 - \alpha_7\mu_4 & \delta_6 - \alpha_7\mu_5 & 0 & 0 \\ \delta_7 - \alpha_8\,{}^t\nu_2 & \delta_8 - \alpha_8\,{}^t\nu_4 & \delta_9 - \alpha_8\,{}^t\nu_6 & 0 & 0 \end{pmatrix}.$$

We have δ_3 square and invertible modulo q; so we need $\delta_5 - \alpha_7\mu_4$ (which is square) to be invertible modulo q. By symmetry, we know $(\delta_5 - \alpha_7\mu_4)\,{}^t\alpha_7$ is symmetric; writing $\mu_4 = \mu_4' + q\mu_4''$ where μ_4', μ_4'' vary over symmetric $d_7 \times d_7$ matrices modulo q, $(\delta_5 - \alpha_7\mu_4')\,{}^t\alpha_7$ does as well. (So there are $q^{d_7(d_7+1)/2}\text{sym}(d_7)$ ways to choose μ_4 so that $\delta_5 - \alpha_7\mu_4$ is invertible modulo q.) So to have B invertible, we need

$$\begin{pmatrix} (\beta_1 - \alpha_5\mu_1)/q & \gamma_1 - \alpha_5\nu_1 & \gamma_2 - \alpha_5\nu_2 \\ 0 & \gamma_3 & 0 \\ \delta_7 - \alpha_8\,{}^t\nu_2 & 0 & 0 \end{pmatrix}$$

to be invertible modulo q. We previously noted that γ_3 is invertible modulo q, so we need

$$\begin{pmatrix} (\beta_1 - \alpha_5\mu_1)/q & \gamma_2 - \alpha_5\nu_2 \\ \delta_7 - \alpha_8\,{}^t\nu_2 & 0 \end{pmatrix}$$

to be invertible modulo q, or equivalently, we need

$$\begin{pmatrix} (\beta_1 - \alpha_5\mu_1)\,{}^t\alpha_5/q & (\gamma_2 - \alpha_5\nu_2)\,{}^t\alpha_5 \\ (\delta_7 - \alpha_8\,{}^t\nu_2)\,{}^t\alpha_8 & 0 \end{pmatrix}$$

to be invertible modulo q, and this latter matrix is symmetric modulo q.

Now we compute $\sum_Y \overline{\chi}_q(M, N)\chi_q(M', N')$. First, we choose a permutation matrix $G_1 \in GL_n(\mathbb{Z})$ so that

$$EM'GG_1 \equiv \begin{pmatrix} A_1' & 0 & 0 & 0 \\ 0 & A_4' & 0 & 0 \\ 0 & 0 & A_5 & 0 \\ 0 & 0 & A_7 & A_8 \end{pmatrix} \ (q),$$

$$EN'\,{}^tG^{-1}\,{}^tG_1^{-1} = \begin{pmatrix} B_1' & B_2' & B_1 & B_2 \\ B_3' & B_4' & B_3 & B_4 \\ B_5' & B_6' & B_5 & B_6 \\ B_7' & B_8' & B_7 & B_8 \end{pmatrix}$$

(recall that since G_1 is a permutation matrix, ${}^tG_1^{-1} = G_1$). Then

$$MG_1 \equiv \begin{pmatrix} A_1' &&& \\ & A_4' && \\ && 0 & \\ &&& 0 \end{pmatrix} \ (q),$$

$$N\,{}^tG_1^{-1} \equiv \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & (B_5 - A_5U_3 - A_6\,{}^tV_4)/q & B_6 - A_5V_4 \\ 0 & 0 & B_7 - A_7U_3 - A_8\,{}^tV_4 & 0 \end{pmatrix} \ (q).$$

Then we choose permutation matrices $E_2', G_2' \in GL_{n-d_1-d_4}(\mathbb{Z})$ so that

$$E_2' \begin{pmatrix} A_5 & 0 \\ A_7 & A_8 \end{pmatrix} G_2' \equiv \begin{pmatrix} \alpha_5 &&&& \\ & \alpha_8 &&& \\ && \alpha_7 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \ (q),$$

$$E_2' \begin{pmatrix} (B_5 - A_5U_3 - A_6\,{}^tV_4)/q & B_6 - A_5V_4 \\ B_7 - A_7U_3 - A_8\,{}^tV_4 & 0 \end{pmatrix} {}^t(G_2')^{-1}$$
$$\equiv \begin{pmatrix} (\beta_1 - \alpha_5\mu_1)/q & \gamma_2 - \alpha_5\nu_2 & * & * & * \\ \delta_7 - \alpha_8\,{}^t\nu_2 & 0 & * & 0 & * \\ 0 & 0 & \delta_5 - \alpha_7\mu_4 & 0 & 0 \\ 0 & 0 & * & \gamma_3 & 0 \\ 0 & 0 & 0 & 0 & \delta_3 \end{pmatrix} \ (q).$$

Set $E_2 = \begin{pmatrix} I_{d_1+d_4} & \\ & E_2' \end{pmatrix}$, $G_2 = \begin{pmatrix} I_{d_1+d_4} & \\ & G_2' \end{pmatrix}$. Then

$$\begin{aligned} \chi_q(\det(E_2G_1G_2))\chi_q(M', N') &= \chi_q(E_2EM'GG_1G_2, E_2EN'\,{}^t(GG_1G_2)^{-1}) \\ &= \overline{\chi}_q(\det A_1' \cdot \det A_4' \cdot \det \alpha_5 \cdot \alpha_7 \cdot \det \alpha_8)\chi_q(\det \gamma_3 \cdot \det \delta_3). \end{aligned}$$

On the other hand,

$$\begin{aligned} \chi_q(\det(E_2G_1G_2))\chi_q(M, N) &= \chi_q(E_2MG_1G_2, E_2N^t(G_1G_2)^{-1}) \\ &= \overline{\chi}_q(\det A_1' \cdot \det A_4')\chi_q(\det \gamma_3 \cdot \det \delta_3) \\ &\quad \cdot \chi_q\left(\det \begin{pmatrix} (\beta_1 - \alpha_5\mu_1)/q & \gamma_2 - \alpha_5\nu_2 \\ \delta_7 - \alpha_8\,{}^t\nu_2 & \end{pmatrix} \cdot \det(\delta_5 - \alpha_7\mu_4)\right). \end{aligned}$$

Thus

$$\overline{\chi}_q(M, N)\chi_q(M', N') = \chi_q\left(\det \begin{pmatrix} (\beta_1 - \mu_1\,{}^t\alpha_5)/q & \gamma_2 - \nu_2\,{}^t\alpha_5 \\ \delta_7 - {}^t\nu_2\,{}^t\alpha_8 & 0 \end{pmatrix} \det(\delta_5 - \mu_4\,{}^t\alpha_7)\right);$$

recall that we have already noted that

$$\begin{pmatrix} (\beta_1 - \mu_1\,{}^t\alpha_5)/q & \gamma_2 - \nu_2\,{}^t\alpha_5 \\ \delta_7 - {}^t\nu_2\,{}^t\alpha_8 & 0 \end{pmatrix},\ \delta_5 - \mu_4\,{}^t\alpha_7$$

are symmetric modulo q. Thus

$$\sum_{\mu_1,\mu_2} \chi_q\left(\det\begin{pmatrix} (\overline{\alpha}_5\beta_1 - \mu_1)/q & \overline{\alpha}_5\gamma_2 - \nu_2 \\ \overline{\alpha}_8\delta_7 - {}^t\nu_2 & 0 \end{pmatrix}\det(\overline{\alpha}_7\delta_5 - \mu_4)\right) = \mathrm{sym}_q^\chi(d_5, d_8),$$

and

$$\sum_{\mu_4} \chi_q(\det(\overline{\alpha}_7\delta_5 - \mu_4)) = \mathrm{sym}_q^\chi(d_7).$$

We have seen that μ_2, μ_3 are determined modulo q, but unconstrained further modulo q^2, μ_5, μ_6 are unconstrained modulo q^2, and $\nu_1, \nu_3, \nu_4, \nu_5, \nu_6$ are unconstrained modulo q. Hence there are

$$q^{(j-d_1)(n-d_1-d_4+1)-d_5(j-d_1+d_8+1)-d_7(d_7+1)/2}\mathrm{sym}(d_7)\mathrm{sym}(d_5, d_8)$$

choices for Y so that M, N are integral with $(M, N) = 1$. Hence, having fixed E, G and then summing over those Y that meet the conditions determined above,

$$\sum_Y \overline{\chi}_q(M, N)\chi_q(M', N') = q^{(j-d_1)(n-d_1-d_4+1)-d_5(j-d_1+d_8+1)-d_7(d_7+1)/2}\mathrm{sym}_q^\chi(d_7)\mathrm{sym}_q^\chi(d_5, d_8).$$

To simplify the formula for $A_j(d, t)$, we note that $r = j - d_1 - d_5 + d_8$, $d = d_1 + d_4 = n - \ell$, $d' = d + t$, $t = d_5 + d_7 + d_8$, $d_1 + d_5 + d_7 \le j$, $d_4 + d_8 \le n - j$, and $d_8 \le d_5$. Using this information yields the formula for $a_j(\ell; d_1, d_5, d_8)$. Also, we know $\beta(m, s) = \beta(m, m - s)$, so

$$\begin{aligned}
&\beta(d_1 + d_4 + d_5, d_1)\beta(d', d_1 + d_4 + d_5)\beta(d_4 + d_5, d_4) \\
&\quad= \frac{\mu(n - \ell + d_5, d_1)\mu(n - \ell + t, t - d_5)\mu(n - \ell - d_1 + d_5, d_5)}{\mu(d_1, d_1)\mu(t - d_5, t - d_5)\mu(d_5, d_5)}\,\frac{\mu(t, d_5)}{\mu(t, d_5)} \\
&\quad= \frac{\mu(n - \ell +, d_1 + t)\mu(t, d_5)}{\mu(d_1, d_1)\mu(t, t)\mu(d_5, d_5)} \\
&\quad= \frac{\mu(n - \ell + t, t)\mu(n - \ell, d_1)\mu(t, d_5)}{\mu(t, t)\mu(d_1, d_1)\mu(d_5, d_5)} \\
&\quad= \beta(d + t, t)\beta(d, d_1)\beta(t, d_5).
\end{aligned}$$

This gives us the formula for $A_j(d, t)$, subject to the constraints on the d_i. Taking $0 \le d_1 \le j$, $0 \le d_5 \le j - d_1$, and $0 \le d_8 \le d_5$, the summand in the formula for $A_j(d, t)$ is 0 if the other constraints on the d_i are not met. $\square$

As discussed after Theorem ??, we know we have a basis $\{\widetilde{\mathbb{E}}_\rho\}_\rho$ of simultaneous eigenforms for the space of Eisenstein series of degree n, weight k, square-free level $\mathcal{N}$, and character χ, and these are eigenforms for all Hecke operators $T(p)$, $T_j(p^2)$ where p is any prime. Below we compute the eigenvalues for $T_j(q^2)$ (where, as above, $q|\mathcal{N}$); in later work we compute the eigenvalues for $T(p)$, $T_j(p^2)$ for p any prime not dividing $\mathcal{N}$.

Corollary 4.5. *Let ρ be a multiplicative partition of $\mathcal{N}$, and suppose $\mathbb{E}_\rho \neq 0$. Then with $d = rank_q M_\rho$, for a prime $q|\mathcal{N}$ and $d = rank_q M_\rho$, we have $\widetilde{\mathbb{E}}_\rho|T_j(q^2) = \lambda_{\rho,j}(q^2)\widetilde{\mathbb{E}}_\rho$ where*

$$\lambda_{\rho,j}(q^2) = q^{jd} \sum_{d_1=0}^{j} q^{d_1(2k-2d-j+d_1-1)} \chi_{\mathcal{N}_0}(q^{2d_1}) \chi_{\mathcal{N}_n}(q^{2(j-d_1)}) \beta(d,d_1)\beta(n-d,j-d_1).$$

Proof. By Corollary 4.3 and Theorem 4.4, we know that $\widetilde{\mathbb{E}}_\rho$ is an eigenform for $T_j(q^2)$ with eigenvalue $A_j(d,0)$. In general, with $r = j - d_1 - d_5 + d_8$, and prime $q'|\mathcal{N}/q$ so that $d' = \mathrm{rank}_{q'} M_\rho$, we know $\chi_{q'}^2 = 1$ for $q'|\mathcal{N}/(\mathcal{N}_0\mathcal{N}_n)$ and thus

$$\chi_{q'}(D_{d_1,r} M_\rho D_j^{-1}, D_{d_1,r} D_j) = \begin{cases} \chi_{q'}(q^{d_5-d_8}) & \text{if } 0 < d' < n, \\ \chi_{q'}^2(q^{d_1})\chi_{q'}(q^{d_5-d_8}) & \text{if } d' = 0, \\ \chi_{q'}^2(q^{j-d_1})\chi_{q'}(q^{-d_5+d_8}) & \text{if } d' = n. \end{cases}$$

Since in the sum for $A_j(d,0)$ we have $d_5, d_8 = 0$, the corollary follows. □

L.H. Walling, Department of Mathematics, University Walk, University of Bristol, Bristol BS8 1TW England